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LETTER TO THE EDITOR

Quantum electrodynamic circuit interpretation of the thermal Nyquist noise theorem in the $\omega \gtrsim (k_B T/\hbar)$ limit

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Abstract. Electrical circuits at low temperatures, $T \lessdot (\hbar\omega/k_B)$, require a quantum electrodynamic treatment of voltage as an operator which creates and destroys photons of frequency ω. Here, we consider the physical interpretation of measuring quantum Nyquist voltage noise using an amplifier with a power gain $G(\omega)$ and Schwinger noise temperature $T^*(\omega) \ll T$.

An important problem in the theory of low-temperature electrical circuits is the nature of the quantum electrodynamic limit, i.e. the experimental signatures that voltage must be represented by operators which create and destroy photons (Widom 1979). For example, the question arises of how to measure thermal Nyquist voltage noise in the low-temperature limit $T \lessdot (\hbar\omega/k_B)$, where ω is the noise frequency. Here, voltage must be considered to be a macroscopic quantum operator which is to be fed into a measuring apparatus, say an amplifier and frequency filter, which then produces a larger classical noise output signal. This classical output is measured in the usual manner. The purpose of this work is to present the theory of quantum voltage noise measurements using a notion of amplifier gain and noise temperature due to Schwinger (1960). Although the general theory of quantum Brownian motion, developed by Schwinger, is fairly complex, the answer to the problem at hand can be simplified if the amplifier and frequency filter are viewed as a photon detector with low noise temperature $T^*(\omega) \ll T$.

Consider a single circuit with a geometrical capacitance C and geometrical inductance L, which measure the length scales (in Gaussian units) of the electromagnetic field oscillations for a single photon mode at frequency

$$\omega_0^2 = (c^2/LC). \tag{1}$$

Here, c is the velocity of light. The renormalised propagator for the photon mode, which includes damping, can be written in terms of a frequency dependent inductance (Widom 1980)

$$L(\zeta) = (\omega_0^2 L)/(\omega_0^2 - \zeta^2 - i\zeta\eta(\zeta)), \tag{2}$$

where the Dyson photon self energy $i\zeta\eta(\zeta)$ can be represented by a shunt impedance $Z_s(\zeta)$ across the capacitance, yielding

$$\eta(\zeta)^{-1} = CZ_s(\zeta). \tag{3}$$

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The fluctuation-response theorem for the quantum electrodynamic magnetic flux noise in the inductor reads

$$S_{\Phi\Phi}(\omega) = E_T(\omega)\,\mathrm{Im}\,L(\omega + \mathrm{i}0^+)/\pi\omega, \tag{4}$$

$$E_T(\omega) = (n(\omega)+\tfrac{1}{2})\hbar\omega, \qquad n(\omega) = (\mathrm{e}^{\hbar\omega/k_BT} - 1)^{-1}. \tag{5}$$

The Nyquist theorem for voltage noise across the inductor follows from Faraday's law $V = -\dot{\Phi}/c$ and the relation between impedance and inductance,

$$Z(\zeta) = -(\mathrm{i}\zeta/c^2)L(\zeta). \tag{6}$$

It is

$$S_{vv}(\omega) = E_T(\omega)\,\mathrm{Re}\,Z(\omega + \mathrm{i}0^+)/\pi. \tag{7}$$

The flux noise spectrum near zero frequency is often expressed in terms of energy sensitivity in units of Planck's constant,

$$\hbar\alpha_T = \lim_{\omega\to 0}(S_{\Phi\Phi}(\omega)/L). \tag{8}$$

However, one should not infer (from the definition of α_T) that quantum theory limits the size of α_T to a number at least of order unity. This error is frequent in the literature (Buhrman 1977, Clarke 1979). The thermal noise energy sensitivity $\hbar\alpha_T$ is given by

$$\alpha_T = (L/\lambda)/(cR), \tag{9a}$$

where R is the shunt resistance

$$R = \lim_{\omega\to 0} Z_s(\omega + \mathrm{i}0^+) \tag{9b}$$

and λ is the thermal photon wavelength

$$\lambda = (\pi\hbar c/k_BT). \tag{9c}$$

High shunt resistance ($cR \gg 1$) and/or low temperature ($\lambda \gg L$) can yield $\alpha_T \ll 1$.

The experimental meaning of quantum Nyquist voltage noise, i.e. equation (7) in the $\omega \gtrsim (k_BT/\hbar)$ limit, involves the explicit engineering by which a macroscopic quantum operator is measured by an apparatus. An example is shown in figure 1. If the thermal photon mode circuit were removed from the measuring device, then the quantum

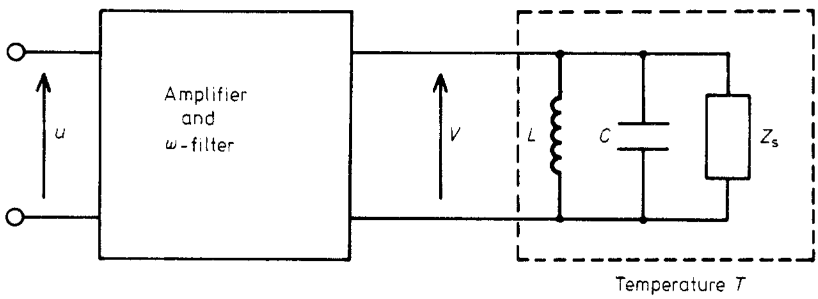

Figure 1. The quantum thermal voltage noise V due to a photon mode circuit is fed into a measuring apparatus. The apparatus amplifies and frequency filters the quantum input ($T \ll \hbar\omega/u_B$), and then produces a classical output noise voltage u. If the apparatus noise temperature $T^* \ll T$, then amplifier feedback into the photon mode circuit can be ignored.

voltage noise would be described by equation (7). The quantum voltage noise can be decomposed into two parts:

$$S_{vv}^{>}(\omega)=\tfrac{1}{2}[S_{vv}^{>}(\omega)+S_{vv}^{<}(\omega)], \qquad (10)$$

$$S_{vv}^{<}(\omega)=\mathrm{e}^{-\hbar\omega/k_{\mathrm{B}}T}S_{vv}^{>}(\omega). \qquad (11)$$

Here, $S_{vv}^{>}(\omega)$ is proportional to the rate at which the quantum photon mode circuit absorbs photons and $S_{vv}^{<}(\omega)$ is proportional to the rate at which the photon mode circuit emits photons.

Similarly, the classical notion of amplifier power gain $G(\omega)$ can be decomposed (Schwinger 1960) in a manner closely analogous to equations (10) and (11),

$$G(\omega)=G^{>}(\omega)+G^{<}(\omega), \qquad (12)$$

where $G^{>}(\omega)$ is proportional to the rate at which the apparatus absorbs photons from the photon mode circuit and $G^{<}(\omega)$ is the rate at which the apparatus emits photons (by feedback) into the photon mode circuit. The Schwinger noise temperature $T^{*}(\omega)$ of the amplifier is defined by

$$G^{<}(\omega)=\mathrm{e}^{-\hbar\omega/u_{\mathrm{B}}T^{*}(\omega)}G^{>}(\omega). \qquad (13)$$

The exchange rate of photons between the amplifier and photon mode circuit is proportional to products of emission and absorption rates:

$$P(\omega)=[G^{>}(\omega)S_{vv}^{<}(\omega)+S_{vv}^{>}(\omega)G^{<}(\omega)], \qquad (14)$$

which depends critically on how the measurement apparatus is power matched to the photon mode circuit.

In the limit of weak amplifier feedback $T^{*}(\omega)\ll T$, equation (14) reads with a sufficient degree of accuracy

$$P(\omega)\simeq G(\omega)S_{vv}^{<}(\omega)=2G(\omega)S_{vv}(\omega)/(1+\mathrm{e}^{\hbar\omega/u_{\mathrm{B}}T}). \qquad (15)$$

This photon exchange rate can be identified with the output power spectrum of classical voltage noise from the measuring apparatus:

$$\bar{S}_{uu}(\omega)=G(\omega)(\hbar|\omega|/\pi)n(|\omega|)\,\mathrm{Re}\,Z(\omega+\mathrm{i}0^{+}) \qquad (16)$$

and

$$\bar{S}_{vv}(\omega)=(\hbar\omega/\pi)[n(\omega)+\tfrac{1}{2}]\,\mathrm{Re}\,Z(\omega+\mathrm{i}0^{+}). \qquad (17)$$

Equations (16) and (17) are principal results of this work on understanding the physical interpretation of quantum electrodynamic Nyquist noise $T\lessdot(\hbar\omega/k_{\mathrm{B}})$ as measured by a low Schwinger noise temperature amplifier $T^{*}(\omega)\ll T$.

The photon operator voltage noise contains the zero-point photon oscillation energy $\frac{1}{2}\hbar\omega$; however, $S_{vv}(\omega)$ is not directly observed. When the operator voltage is fed through the amplifier (measurement apparatus) and the spectrum for the classical output noise $\bar{S}_{uu}(\omega)$, $\omega>0$, is observed with an amplifier gain $G(\omega)$, only the mean number of photons $n(\omega)$ in the photon mode circuit enters into the measured result. In the classical limit $n(\omega)\gg 1$, the obvious classical Nyquist result for an amplifier with gain $G(\omega)$ follows;

$$\bar{S}_{vv}(\omega)=(k_{\mathrm{B}}\mathrm{T}/\pi)\,\mathrm{Re}\,Z(\omega+\mathrm{i}0^{+}), \qquad (18a)$$

$$\bar{S}_{uu}(\omega)=G(\omega)\bar{S}_{vv}(\omega). \qquad (18b)$$

References

Buhrman R A 1977 in *Superconducting Quantum Interference Devices and Their Applications* (Berlin: Walter de Gruyter) pp 395–431

Clarke J 1979 *Phys. Bull.* **30** 207

Schwinger J 1960 *J. Math. Phys.* **2** 407–32

Widom A 1979 *J. Low Temp. Phys.* **37** 449–60

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